Period-multiplying bifurcations and multifurcations in conservative mappings



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# Period-multiplying bifurcations and multifurcations in conservative mappings

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**Abstract.** We have investigated numerically and analytically the period-doubling bifurcations and the multifurcations of the periodic orbits of the conservative sine-Gordon mappings. We have derived a general equation for the appearance of multifurcations in conservative mappings. In agreement with many recent studies, we also find evidence that such mappings possess universality properties. We also discuss the role of multifurcations in conservative mappings exhibiting chaotic behaviour.

## 1. Introduction

A recent theory by Feigenbaum (1978) of nonlinear systems which exhibit period doublings approach to chaos predicts that these systems behave in a universal manner independent of the precise equations which govern their dynamics. In dissipative systems the bifurcations of periodic solutions of the particular differential or difference equations have provided a possible mechanism for the onset of chaotic or turbulent behaviour (Libchaber and Mauer 1980).

In this investigation we study analytically and numerically bifurcations and multifurcations of periodic orbits of a conservative system. In particular we study the orbits of a general period $N$ of the sine-Gordon mapping

$$T: \quad \begin{aligned} x_{n+1} &= -y_n + 2x_n + pv \sin px_n \\ y_{n+1} &= x_n \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{1.1}$$

with $p = 4$. The mapping (1.1) is an area-preserving transformation (its Jacobian is equal to $+1$) and generates in the $(x, y)$ plane pictures similar to the surfaces of section of Hamiltonian systems with two degrees of freedom (Fradkin and Huberman 1981).

The mapping (1.1) is based on the familiar sine-Gordon equations, which in the continuum form are of the type

$$\ddot{\phi} - pv \sin p\phi = 0. \tag{1.2}$$

These equations govern the dynamics of several nonlinear systems in classical mechanics and in condensed matter physics, since they correspond to the energy extremum conditions of a variety of systems. Writing equation (1.2) in the difference form on a discrete lattice with equally spaced spins, atoms, etc, we obtain

$$(\phi_{n+1} - \phi_n) - (\phi_n - \phi_{n-1}) - pv \sin p\phi_n = 0. \tag{1.3}$$

To see that equation (1.3) indeed gives a two-dimensional area-preserving mapping

 

(1.1), let us introduce new variables $x_n$ and $y_n$ defined by (Yamaguti and Ushiki 1981)

$$x_n = \phi_n \qquad y_n = \phi_{n-1}. \tag{1.4}$$

The transformation (1.3) now takes the form (1.1). Bak and Jensen (1982) have discussed the connection between the behaviour in the discrete models and in the continuous systems. Their analysis shows that this connection is not trivial. The discrete mathematical models apply, however, almost directly to problems related to solids, because in condensed matter physics the physical quantities and fields are often defined on a discrete lattice.

The properties of structurally and magnetically modulated systems have been analysed numerically and analytically by several authors (Aubry 1980, Bak 1981, Bak and Pokrovsky 1981). It was found that chaotic structures are at least metastable (Yorke and Yorke 1979). We study the onset of the chaotic behaviour through both ordinary bifurcations and multifurcations in these models. On the basis of the numerical calculations we determine the Feigenbaum convergence number

$$\delta = \lim_{k\to\infty} \frac{v_{k-2} - v_{k-1}}{v_{k-1} - v_k} \tag{1.5}$$

where $v_k$ is the value of the (control) parameter $v$, for which the bifurcation from period $2^k$ to period $2^{k+1}$ takes place in the mapping (1.1). We find that, up to our numerical accuracy, $\delta$ seems to converge towards the value $\delta = 8.721\,09\ldots$ found also in other conservative systems (Benettin *et al* 1980a, b, Bountis 1981, Greene *et al* 1981, Bak and Jensen 1982). The first three bifurcations of the mapping have also been studied analytically. In addition to the infinite series of bifurcations we find the feature which has been seen by Bak and Jensen (1982) in their recent study of $\phi^4$ lattice theory: a two-cycle splits into two two-cycles and not into one four-cycle as one would expect at a regular bifurcation.

In conservative two-dimensional mappings, the emergence of multifurcations (Birkhoff multifurcations) is a very familiar feature. Multifurcation normally produces $N$ stable elliptic fixed points (FPS) and also $N$ unstable hyperbolic FPS. This is important, because chaotic behaviour concentrates into the vicinity of hyperbolic FPS. In this respect multifurcations in addition to ordinary bifurcations play a major role in creating chaos. Multifurcations do not tend to emerge in a random way during the route of ordinary bifurcations to chaos, but their appearance follows, according to our study, strict conditions characterised by Chebyshev polynomials. In particular, we develop a systematic theory for the conditions of multifurcated FPS to appear in conservative area-preserving mappings. We find that each $Q$-cycle FP of the mapping (1.1) experiences an $N$-furcation of its own according to the law

$$\mathrm{Tr}\, M = \mathrm{Tr}\left( \prod_{n=1}^{Q} m_n \right) = 2\cos\frac{2k\pi}{N} \tag{1.6}$$

where $N$ can be regarded as a primary winding number and $k$ as a secondary one. The matrix

$$\mathbf{m}_n = \begin{pmatrix} \delta x_{n+1}/\delta x_n & \delta x_{n+1}/\delta y_n \\ \delta y_{n+1}/\delta x_n & \delta y_{n+1}/\delta y_n \end{pmatrix} \tag{1.7}$$

relates the tangent space orbits at the points $(x_n, y_n)$ and $(x_{n+1}, y_{n+1})$:

$$(\delta x_{n+1}, \delta y_{n+1})^{\mathrm{T}} = \mathbf{m}_n (\delta x_n, \delta y_n)^{\mathrm{T}}. \tag{1.8}$$

Det $\mathbf{m}_n$ is the Jacobian of the transformation and is equal to +1 for the mapping (1.1).

On the basis of equation (1.6) we investigate analytically, and also by numerical computations, the various multifurcations of the sine-Gordon mapping. We find that our numerical study and our graphical pictures fit equation (1.6) amazingly well. In addition to an infinite series of multifurcations of the $Q = 2^k$-cycle FPS, we find that also multifurcated FPS undergo multifurcations and series of bifurcations exhibiting the universal Feigenbaum number for conservative systems. We also find anomalous multifurcations for $N = 3$ and $N = 4$. In § 2 we discuss the period doublings and the stability analysis of sine-Gordon mappings. Section 3 contains a general theory for multifurcations in conservative systems. This is applied to the sine-Gordon equation in § 4; a short summary is given in § 5.

## 2. Period doublings and stability analysis

In this section we obtain analytically the first three period-doublings of the mapping

$$T: \qquad \begin{aligned} x_{n+1} &= -y_n + 2x_n + 4v \sin 4x_n \\ y_{n+1} &= x_n. \end{aligned} \tag{2.1}$$

We also analyse the stability properties of the fixed points. Then following numerically (and also graphically) the behaviour of the mapping near these periodic orbits, as the control parameter $v$ is increased, we obtain useful information about the higher-order orbits of the periodic-doubling sequence: $Q = 2^k$, for $k = 0, 1, 2, \ldots$. We start to consider the iterates of (2.1) in the $(x_n, y_n)$ plane. Let $\{x_n, x_{n+Q} = x_n\}$ denote a $Q$-period orbit of (2.1). A periodic orbit of the map is a finite sequence of distinct points, each of which is the image of the previous one, and whose first point is the image of the last. Following the usual stability methods (Greene 1979) we can say that the periodic orbit of period $Q$ (or $Q$-cycle) is stable or unstable according to whether each of its points is stable or unstable when considered as a fixed point of $T^Q$. By continuity, they are all stable or unstable together, so it is sufficient to examine the stability of one of them. Linearising $T^Q$ about $\{x_n\}$ we obtain a $2\times 2$ matrix $\mathbf{M}$ which governs the motion of nearby points $\{x_n + \xi_n\}$ according to

$$\bar{\xi}' = \mathbf{M}\bar{\xi} + \mathrm{O}(\bar{\xi}^2). \tag{2.2}$$

The stability of the point under $T^Q$ is largely determined by the eigenvalues $\lambda$ of $\mathbf{M}$, called the multipliers of the periodic orbit. By the chain rule, $\mathbf{M}$ can be written as the product of the matrices $\mathbf{m}_n$ (equations (1.7) and (1.8)) at each point around the orbit, starting at the given point. As $\mathbf{m}_n$ has determinant +1 everywhere to preserve area, $\mathbf{M}$ must also have determinant 1. Together with the reality of $\mathbf{M}$, this limits the eigenvalues to conjugate points $\lambda$, $\bar{\lambda}$ on the unit circle in the complex plane, or reciprocal points $\lambda$, $1/\lambda$ on the real axis. In the latter case the periodic points are clearly unstable, but in the former case they can be shown to be stable (Greene 1979). The eigenvalues $\lambda$ of $\mathbf{M}$ depend only on its trace, Tr $\mathbf{M}$, and are the roots of

$$\lambda^2 - \mathrm{Tr}(\mathbf{M})\lambda + 1 = 0. \tag{2.3}$$

Thus the stability conditions are:

$$|\mathrm{Tr}\,\mathbf{M}|\begin{cases}<2 & \text{stable or elliptic}\\ >2 & \text{unstable or hyperbolic}\\ =2 & \text{marginally (un)stable or parabolic}\end{cases} \tag{2.4}$$

where

$$\mathbf{M}=\prod_{n=1}^{Q}\mathbf{m}_n \tag{2.5}$$

for $Q$-periodic orbit $\{x_n\}$.

We start to consider the iterates of equation (2.1) in the $(x_n, y_n)$ plane. We linearise equation (2.1) about $\{x_n\}$ and write the resulting variational equations in matrix form:

$$(\delta x_{n+1}, \delta y_{n+1})^{\mathrm{T}} = \mathbf{m}_n(\delta x_n, \delta y_n)^{\mathrm{T}} \tag{2.6}$$

$$\mathbf{m}_n = \begin{pmatrix} 2+16v\cos 4x_n & -1 \\ 1 & 0 \end{pmatrix}.$$

Det $\mathbf{m}_n$ is equal to $+1$, which confirms that the mapping is indeed area preserving. With the help of equations (2.1)–(2.6) we find two fixed points of period one of the mapping: (*a*) $(x, y) = (\frac{1}{4}\pi, \frac{1}{4}\pi)$ and (*b*) $(x, y) = (0, 0)$. The choice (*b*) gives an unstable FP for all $v > 0$. The choice (*a*), on the other hand, has $|\mathrm{Tr}\,\mathbf{M}| \leqslant 2$ for $0 \leqslant v \leqslant 0.25$. It can be noted that the period-one solution (one-cycle) is independent of $v$ over the range $0 < v < 0.25$. Figure 1 illustrates graphically the iterates of the mapping for $v = 0.15$. We follow this one-cycle FP until its $|\mathrm{Tr}\,\mathbf{M}| = 2$, at which point it turns unstable or hyperbolic yielding the stable orbit of twice the period.

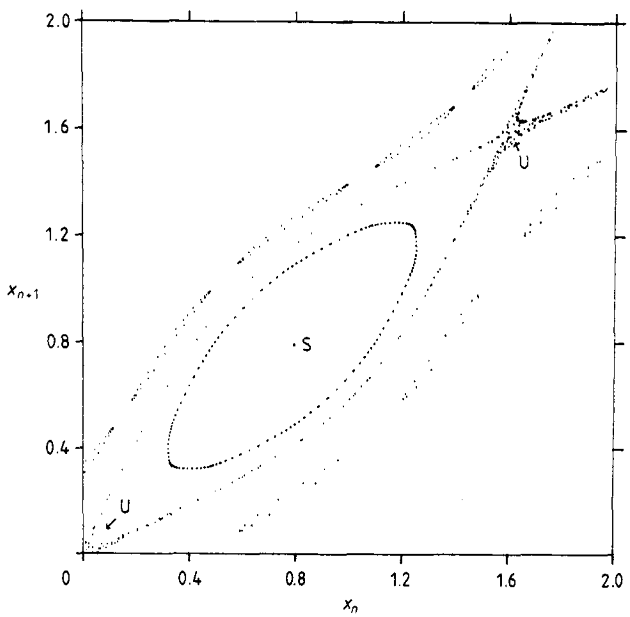


**Figure 1.** Orbits of the sine-Gordon mapping (2.1) at $v = 0.15$ near the stable fixed point $(\frac{1}{4}\pi, \frac{1}{4}\pi)$ surrounded by closed KAM surfaces. The FPs are denoted by S (stable) and U (unstable).

At $v = 0.25$ the first such periodic doubling occurs, as shown in figures 2($a$) and ($b$). The islands around the one-cycle become thinner and rotate 90° compared with figure 1. After the bifurcation to two-cycle the two stable FPs begin to move in opposite directions. The one-cycle FP $(\frac{1}{4}\pi, \frac{1}{4}\pi)$ becomes unstable or hyperbolic with the reflection and a chaotic sea concentrates in its vicinity. The two new elliptic FPs are surrounded by closed Kolmogorov–Arnold–Moser (KAM) orbits of their own. In the iteration procedure, the two islands are visited successively. After the bifurcation there are still closed KAM orbits encircling both FPs. At larger values of $v$ these orbits will disappear (Greene 1979). A plot of the fixed points $x_n$ against $v$ over the range $0 < v < 0.414\,65$ are shown in figures 3($a$) and ($b$). It can be seen that the two branches of the two-cycle FPs are symmetric with respect to $\frac{1}{4}\pi$, so that $x_1 = \frac{1}{2}\pi - x_2$.

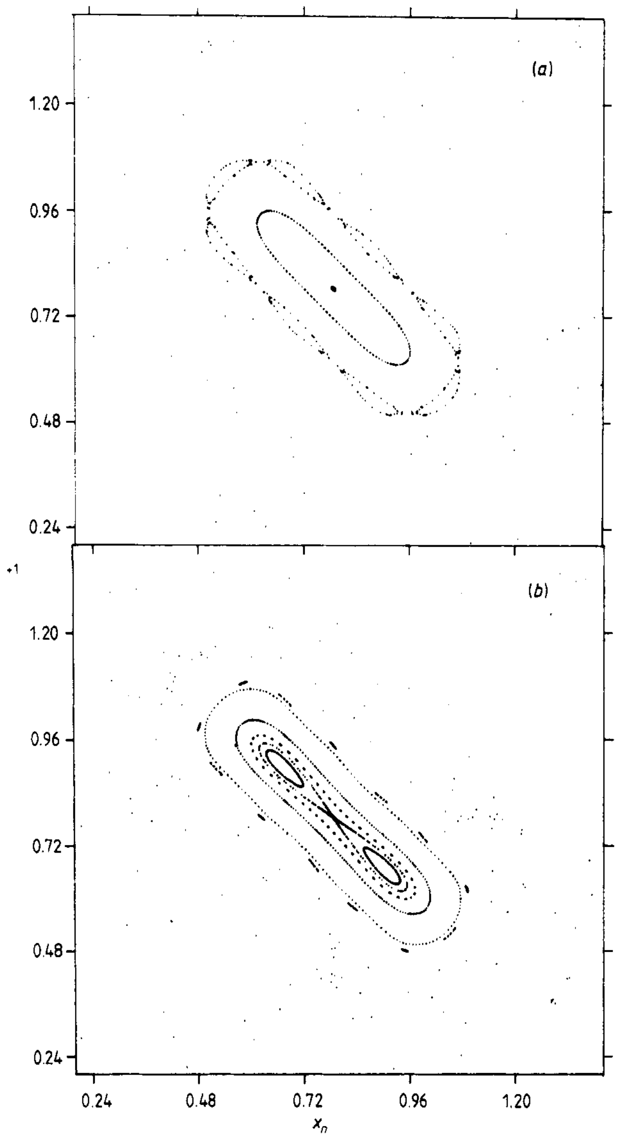


**Figure 2.** ($a$) Orbits of the mapping (2.1) at $v = 0.247$ before the period-doubling bifurcation of period 1–2. Note that the KAM surfaces have rotated 90° compared with figure 1. The stable FP $(\frac{1}{4}\pi, \frac{1}{4}\pi)$ is also surrounded by eight Birkhoff islands. ($b$) Orbits at $v = 0.253$ after the first period-doubling bifurcation. The original stable FP has turned unstable and two new stable FP have emerged.

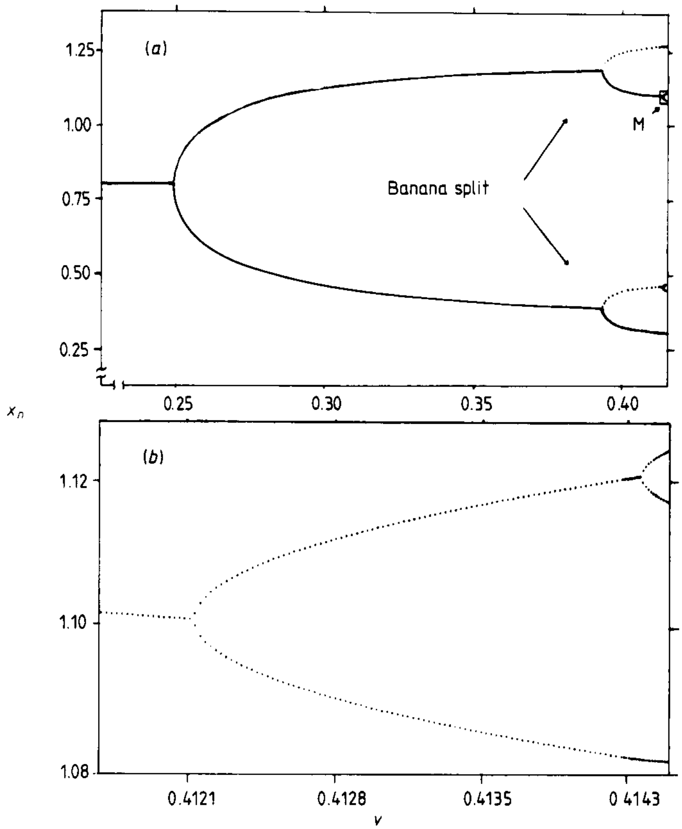


**Figure 3.** ($a$) The bifurcation tree: a plot of the period-doubling FPs as a function of the control parameter $v$ over the range $0 < v < 0.414\,65$. An anomalous bifurcation ('banana split') to $(2\times 2)$-cycle is indicated. Note that bifurcations may produce 'forks' in the $(y_n, v)$ plane instead of the $(x_n, v)$ plane. The dotted curve stresses the split nature of the bifurcation. ($b$) Magnification of the area denoted by M in ($a$).

The two-cycle FPs become unstable at the point $(x_1, y_1) = (\frac{1}{8}\pi, \frac{3}{8}\pi)$, $(x_2, y_2) = (\frac{3}{8}\pi, \frac{1}{8}\pi)$ when $v = \frac{1}{8}\pi$. This can be calculated analytically by taking advantage of certain symmetries of the mapping. Figures 4($a$) and ($b$) show the orbits calculated before and after the bifurcation at $v = 0.38$ and $v = 0.40$. Half of the points are shown. We find that the FPs split into two elliptic FPs with a hyperbolic FP in between. At the bifurcation point $\mathrm{Tr}\,\mathbf{M} = +2$. Hence, the original two-cycle FPs turn to ordinary hyperbolic ones. The new FPs belong to two different and symmetric two-cycles surrounded by KAM orbits. This unusual behaviour is called the 'banana split' and has been found very recently by Bak and Jensen (1982) in their study of $\phi^4$ lattice theory. This anomalous phenomenon is also illustrated in figure 3($a$).

These two two-cycles become unstable at $v = \frac{1}{8}(\pi^2+1)^{1/2} \simeq 0.412\ldots$. This point can also be calculated analytically. The real bifurcation to the four-cycle happens at the point $(\frac{1}{4}\sin^{-1}(\pi/(\pi^2+1)^{1/2}), \frac{1}{2}\pi + \frac{1}{4}\sin^{-1}(\pi/(\pi^2+1)^{1/2}))$ for the former two-cycle and at the point $(\frac{1}{4}\pi - \frac{1}{4}\sin^{-1}(\pi/(\pi^2+1)^{1/2}), \frac{1}{2}\pi + \frac{1}{4}\sin^{-1}(\pi/(\pi^2+1)^{1/2}))$ for the latter two-cycle. Over the whole range $\frac{1}{8}\pi < v < \frac{1}{8}(\pi^2+1)^{1/2}$ the distance between the two successive iterates of the mapping is equal to $\frac{1}{4}\pi$ for both two-cycles.

The FPs belonging to higher-order cycles cannot be found easily analytically. We therefore resort to numerical computations. Even so, locating the initial conditions of a periodic orbit in the plane can become quite tedious if one has to search in both $x$ and $y$ directions. A method developed by Greene (1979) makes the numerical search easier. Greene's method shows that two of the FPs, belonging to a $2N$-cycle, are on a simple one-dimensional curve in the $xy$ plane. The rest of the points follow automatically by iteration.

The mapping (2.1) can be decomposed into a product of two involutions (Bountis 1981, Bak and Jensen 1982):

$$
\begin{aligned}
&I = I_2 I_1 \\
&I_1\colon \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x_n \\ -y_n + 2x_n + 4v \sin 4x_n \end{bmatrix} \\
&I_2\colon \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} y_n \\ x_n \end{bmatrix}
\end{aligned}
\tag{2.7}
$$

where $I_1^2 = I_2^2 = 1$. Now the periodic orbits of $T$ can be found by choosing the initial conditions $(x_1, y_1)$ which are FPs of $I_1$:

$$(x_1, y_1) = I_1(x_1, y_1) \tag{2.8}$$

or

$$2y_1 = 2x_1 + 4v \sin 4x_1. \tag{2.9}$$

Together with $(x_1, y_1)$ this curve also contains the point $(x_m, y_m)$, $m = 2^{k-1}$, of every $2^k$-periodic orbit. The curve (2.9) is called the dominant symmetry curve of the mapping (2.1).

With the help of Greene's method we followed numerically the bifurcation route to chaos. In numerical computations we used 16-digit accuracy. We found that successive bifurcations occurred at $v_4 = 0.414\,362\,8\ldots$, $v_5 = 0.414\,620\,9\ldots$, $v_6 = 0.414\,650\,4\ldots$, $\ldots$, $v_\infty = 0.414\,653\,8\ldots$. These numerical results give the value $\delta = 8.721\,09$ for the Feigenbaum constant (equation (1.5)). Our result agrees well with the value of $\delta$ found in other conservative systems (Bountis 1981, Greene *et al* 1981, Bak and Jensen 1982).

## 3. Multifurcations in conservative mappings

The appearance of Birkhoff islands or $N$-cycle orbits is one of the most common features in conservative two-dimensional area-preserving mappings. No systematic study of the onset of general $N$-cycle orbits has been reported so far. In the following we make an attempt to do so.

Consider a conservative area-preserving mapping of a general type

$$x_{n+1} + bx_{n-1} = F(x_n, v) \qquad b = \pm 1 \tag{3.1}$$

which can be transformed with the help of equation (1.4) into the two-dimensional form

$$x_{n+1} = -by_n + F(x_n, v) \qquad y_{n+1} = x_n. \tag{3.2}$$

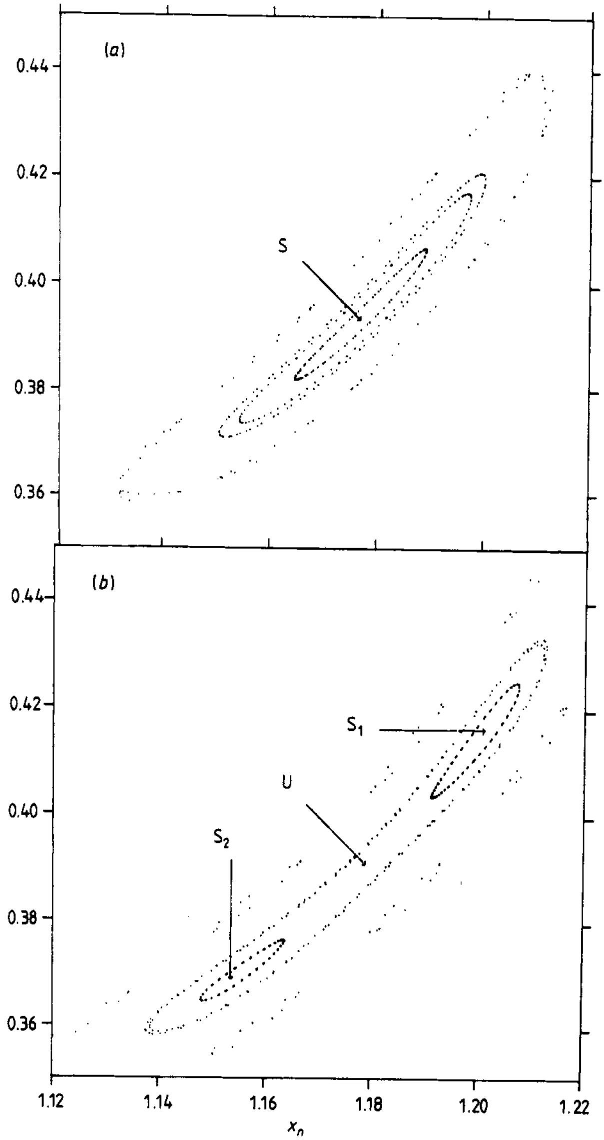


**Figure 4.** (*a*) The 'banana' ($v = 0.38$). The two-cycle is approaching an instability ($v = \frac{1}{8}\pi$). (*b*) The 'banana split' ($v = 0.40$). The two-cycle has not actually bifurcated, but has split into two different two-cycles indicated by $S_1$ and $S_2$. Half of the points are shown.

The stability of a $P$-cycle (note, $P$ may be different from $2^k$, § 4) orbit $(x_1, x_2, \ldots, x_P)$ depends on the matrix $\mathbf{M}$,

$$(\delta x_P, \delta y_P)^{\mathrm{T}} = \mathbf{M}(\delta x_1, \delta y_1)^{\mathrm{T}}$$
$$\mathbf{M} = \prod_{n=1}^{P} \mathbf{m}_n = \prod_{n=1}^{P} \begin{pmatrix} F'(x_n, v) & -b \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

The $P$-cycle is stable if $|\mathrm{Tr}\,\mathbf{M}| < 2$. For the one-cycle we obtain directly $|\mathrm{Tr}\,\mathbf{M}| = |F'(x_1, v)| < 2$.

Suppose that at a certain value of parameter $v$ the one-cycle FP is surrounded by $N$ Birkhoff islands. Thus the condition for the Birkhoff $N$-cycle to be stable

$$|\mathrm{Tr}\,\mathbf{M}| = \left|\mathrm{Tr} \prod_{n=1}^{N} \mathbf{m}_n\right| < 2 \tag{3.4}$$

is fulfilled. Let us now study the brief history of the stable $N$-cycle by decreasing the value of the control parameter $v$. Each of the $N$-cycle FPS, $x_1, \ldots, x_N$, tends to converge toward the one-cycle stable 'mother' orbit. In the limit where the $N$-cycle orbits are born, we can denote $x_1 = x_2 = \ldots = x_N$. The $N$-cycle has become marginally stable. This is obvious, since on the other hand nothing prevents us regarding the one-cycle as an $N$-cycle. The limiting case is covered by the condition

$$|\mathrm{Tr}\,\mathbf{M}| = \left|\mathrm{Tr}\prod_{n=1}^{N}\mathbf{m}_n\right| = |\mathrm{Tr}\,\mathbf{m}^N| = 2 \tag{3.5}$$

where

$$\mathbf{m}^N = \begin{bmatrix} F'(x_1, v) & -b \\ 1 & 0 \end{bmatrix}^N.$$

In calculating the trace of $\mathbf{M}$ we can take advantage of the property of square matrices:

$$\mathrm{Tr}\,\mathbf{m}^N = \lambda_1^N + \lambda_2^N \tag{3.6}$$

where $\lambda_{1,2}$ are the two eigenvalues of the matrix $\mathbf{m}$. For simplicity we denote $d_n = F'(x_n, v)$. The eigenvalues $\lambda_{1,2}$ are found by solving the corresponding characteristic equation

$$\lambda^2 - d_1\lambda + b = 0 \tag{3.7}$$

which gives

$$\lambda_{1,2} = \tfrac{1}{2}d_1 \pm [(\tfrac{1}{2}d_1)^2 - b]^{1/2}. \tag{3.8}$$

For $\mathrm{Tr}\,\mathbf{m}^N$ we obtain

$$\begin{aligned} \mathrm{Tr}\,\mathbf{m}^N &= \{\tfrac{1}{2}d_1 + [(\tfrac{1}{2}d_1)^2 - b]^{1/2}\}^N + \{\tfrac{1}{2}d_1 - [(\tfrac{1}{2}d_1)^2 - b]^{1/2}\}^N \\ &= b^{N/2}(\{\tfrac{1}{2}b^{-1/2}d_1 + [(\tfrac{1}{2}b^{-1/2}d_1)^2 - 1]^{1/2}\}^N \\ &\quad + \{\tfrac{1}{2}b^{-1/2}d_1 - [(\tfrac{1}{2}b^{-1/2}d_1)^2 - 1]^{1/2}\}^N) \\ &= 2b^{N/2}T_N(\tfrac{1}{2}b^{-1/2}d_1) \end{aligned} \tag{3.9}$$

where $T_N$ is the $N$th-order Chebyshev polynomial of the first kind. We can now write the condition (3.5) for an $N$-cycle to emerge in the form

$$|\mathrm{Tr}\,\mathbf{M}| = |\mathrm{Tr}\,\mathbf{m}^N| = |2(b^{1/2})^N T_N(\tfrac{1}{2}d_1 b^{-1/2})| = 2. \tag{3.10}$$

For $b = +1$ we obtain directly

$$|\mathrm{Tr}\,\mathbf{M}| = |2T_N(\tfrac{1}{2}d_1)| = |C_N(d_1)| = 2 \tag{3.11}$$

where $C_N$ is the $N$th-order Chebyshev polynomial of the second kind. Taking advantage of the properties of Chebyshev polynomials the roots of equation (3.11) can be written down immediately as

$$d_1 = 2\cos(2k\pi/N) \tag{3.12}$$

where $N$ can be regarded as the primary and $k$ as the secondary winding number. Recalling that for the one-cycle

$$d_1 = F'(x_1, v) = \mathrm{Tr}\,\mathbf{M}, \tag{3.13}$$

we can finally write equation (3.12) in the form

$$\mathrm{Tr}\,\mathbf{M} = 2\cos(2k\pi/N). \tag{3.14}$$

Equation (3.14) can be generalised to deal with the appearance of multifurcations of general $P$-cycle mother orbits. If we denote

$$\mathbf{M} = \prod_{n=1}^{P} \mathbf{m}_n \tag{3.15}$$

then equation (3.14) gives the condition for $(P \times N)$-furcation to appear, that is by increasing the parameter $v$, each of the $P$-cycle FPs are surrounded by $N$ Birkhoff islands. The iteration procedure follows the $(P \times N)$-furcated FPs successively in a manner where FPs belonging to each mother orbit ($P$-cycle) are visited in turn. If we study the history of the $(P \times N)$-cycle, we find that in the limiting case the $(P \times N)$-cycle fixed points $x'_1, x'_{1+P}, \ldots, x'_{NP-P}$; $x'_2, x'_{2+P}, \ldots, x'_{NP-P+1}$; $\ldots$; $x'_P, x'_{2P}, \ldots, x'_{NP}$ tend to converge toward the $P$-cycle fixed points $x_1, x_2, \ldots, x_P$, respectively. So we obtain for $(P \times N)$-cycle

$$\mathbf{M} = \prod_{n=1}^{P\times N} \mathbf{m}_n = \left[\prod_{n=1}^{P} \mathbf{m}_n\right]^N. \tag{3.16}$$

Furthermore, it can be easily shown (Greene *et al* 1981) that the eigenvalues of the matrices $\Pi_{n=1}^{P}\, \mathbf{m}_n$ can be written in the form

$$\lambda_{1,2} = \tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{M} \pm [(\tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{M})^2 - 1]^{1/2} \tag{3.17}$$

where $\mathrm{Tr}\,\mathbf{M}$ is the trace of $\Pi_{n=1}^{P}\, \mathbf{m}_n$. With the aid of the previous results (equations (3.6), (3.8)–(3.10), (3.16)–(3.17)) with $b = +1$ we can write the condition for very general $(P \times N)$-furcation to appear in the form (3.14):

$$\mathrm{Tr}\,\mathbf{M} = 2\cos(2k\pi/N) \tag{3.18}$$

where $\mathrm{Tr}\,\mathbf{M}$ is now the trace of the matrix $\mathbf{M}$ of the $P$-cycle mother orbit.

## 4. Multifurcations in the sine-Gordon system

In this section we apply the theory developed in the previous section to the multifurcations of the mapping (2.1). At the beginning we investigate the multifurcations before the appearance of the first period-doubling bifurcation. In this case we are fortunate since the one-cycle stable FP $(\tfrac{1}{4}\pi, \tfrac{1}{4}\pi)$ is constant over the whole range $0 < v < 0.25$. The matrix $\mathbf{M}$ of the mapping (2.1) is

$$\mathbf{M} = \begin{pmatrix} 2-16v & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

The multifurcations to the $N$-cycles are possible whenever

$$2 - 16v = 2\cos(2k\pi/N) \tag{4.2}$$

or

$$v = \tfrac{1}{8}[1 - \cos(2k\pi/N)]. \tag{4.3}$$

We find that for the values $2k/N = 0$, $1$ the equation (4.3) gives the limits of stability of the one-cycle: $v = 0$, $v = \tfrac{1}{4}$. A three-furcation is expected for $2k/N = \tfrac{2}{3}$ or when $v = 0.1875$. $\mathrm{Tr}\,\mathbf{M}$ of the mother orbit (one-cycle) is equal to $-1$ at that point. This point is a special one for the three-cycle. Our numerical calculations show that two different three-furcations take place at that point. The orbits at $v = 0.19$ after the three-furcations are shown in figure 5(*a*). Both of the three-cycles are stable. This partly supports the analysis of Arnold (1978) and Meyer (1970). They also explain the anomalous behaviour of branches with $N = 3$, which originally is due to the fact that the linearisation of the mapping (2.1) is not sufficient to describe the behaviour of three-furcations. A four-furcation is possible for $2k/N = \frac{2}{4}$. The ratio $2k/N = \frac{2}{4} = \frac{1}{2}$ is really expected, because it was not used up by an ordinary bifurcation. The value of the parameter $v$ for a four-furcation can be calculated immediately: $v = 0.125$. Tr $\mathbf{M}$ of the mother orbit at that point is equal to zero. In figure 5(*b*) the orbits after a four-furcation are shown. Meyer (1970) also predicts the anomalous behaviour of the four-furcations, at least when Tr $\mathbf{M} = 0$ for the mother orbit. We find, however, that an ordinary four-furcation takes place exactly at the point $v = 0.125$ leading to four stable and four unstable FPs. Moreover, we find five-furcations at $2k/N = \frac{2}{5}$ and $2k/N = \frac{4}{5}$. Hence, the only possibility for a six-furcation to occur is at $2k/N = \frac{2}{6}$, since

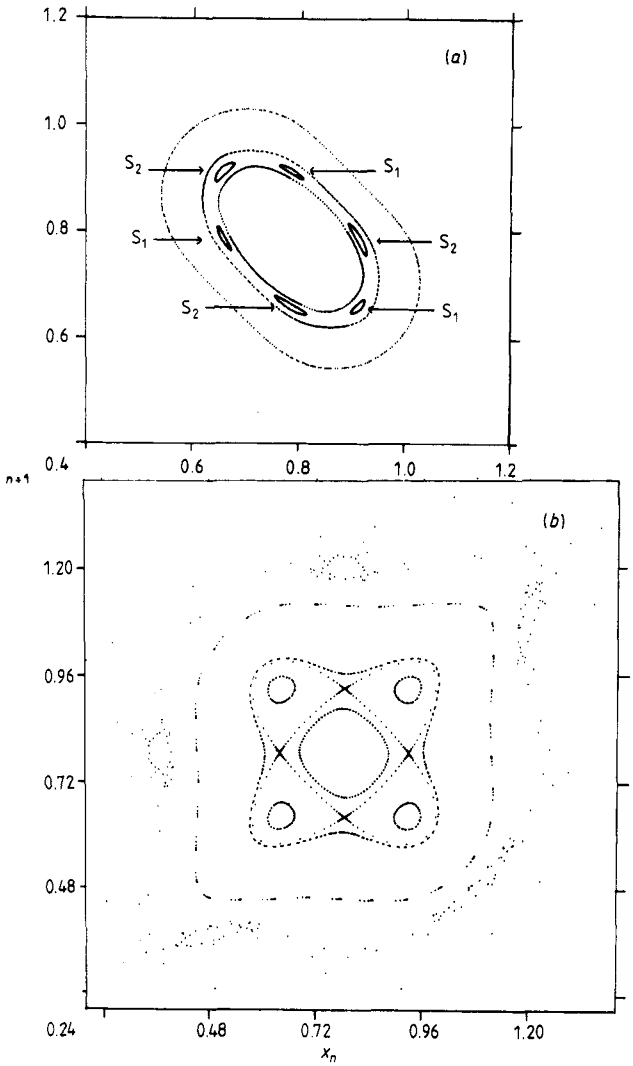


**Figure 5.** (*a*) Orbits of the mapping at $v = 0.19$ after an anomalous three-furcation of the one-cycle. Two different stable three-cycles have emerged. They are indicated by $S_1$ and $S_2$. Six unstable FPs (unstable six-cycle) lie between the stable FPs of the three-cycles. Note that the one-cycle FP and both the three-cycle FPs are surrounded by the common KAM surfaces. (*b*) Orbits of the mapping at $v = 0.13$ after an ordinary four-furcation of the one-cycle. Note that the connection between five Birkhoff islands (five-furcated FPs) has already disappeared while the four-furcated FPs are still surrounded by the common KAM surfaces.

the ratio $2k/N=\frac{4}{6}=\frac{2}{3}$ is used up by three-furcations. One can proceed in this way up to higher-order multifurcations. For example, an ordinary eight-furcation is illustrated in figure 2(*a*).

In particular, we followed numerically one of the four-furcated islands on its way to chaos. We find that it exhibits the Feigenbaum convergence number $\delta=8.721\ldots$ for conservative systems. Similar results have previously been obtained by Bak and Jensen (1982) in their study of $\phi^4$ theory. We find also that the $N$-furcated FPs experience multifurcations. Together with ordinary bifurcations this phenomenon is important in creating chaos. The orbits of one of the four-furcated islands and its five-furcation ((4×5)-cycle) are shown in figure 6. The first bifurcation ((4×2)-cycle) is illustrated in figures 7(*a*) and (*b*).

Multifurcations in two-cycle mother orbits can be studied only by numerical computations. As previously mentioned, $x_1=\frac{1}{2}\pi-x_2$ for a two-cycle. With the aid of equation (2.1) we obtain

$$2x_1=2(\tfrac{1}{2}\pi-x_1)+4v\sin 4(\tfrac{1}{2}\pi-x_1)$$

or

$$v=\tfrac{1}{4}(\pi-4x_1)/(\sin 4x_1) \tag{4.4}$$

and

$$\mathrm{Tr}\,\mathbf{M}=\mathrm{Tr}\prod_{i=1}^{2}\begin{vmatrix}2+16v\cos 4x_n & -1\\ 1 & 0\end{vmatrix}=(2+16v\cos 4x_1)^2-2.$$

The condition for $N$-furcations ($(2\times N)$-cycles) will give

$$v=\left(\frac{-2\pm 2[\cos(2k\pi/N)+1]}{16\cos 4x_1}\right)^{1/2}. \tag{4.5}$$

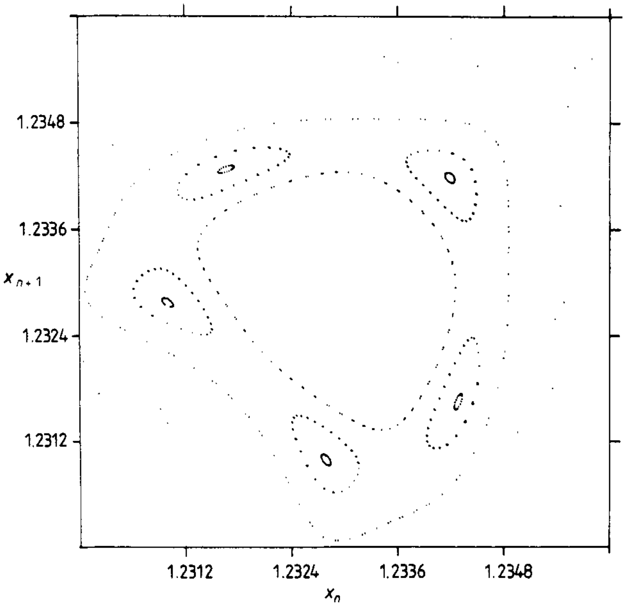


**Figure 6.** Orbits near one of the four-furcated FPs at $v=0.2012$ after the five-furcation of period 4–20.

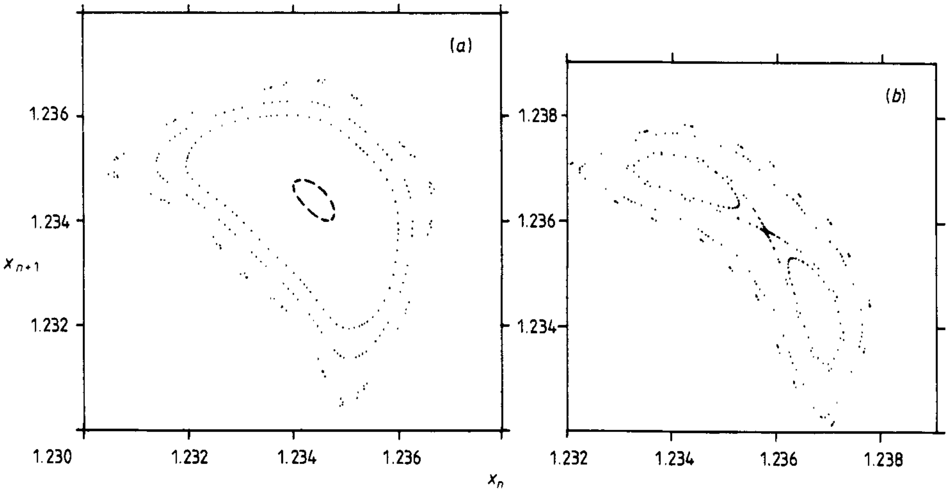


**Figure 7.** Orbits near the four-furcated FP ($a$) just before a period-doubling bifurcation at $v = 0.2014$ and ($b$) after bifurcation to the $(4\times 2)$-cycle at $v = 0.2017$.

The coupled pair of equations (4.4) and (4.5) cannot be solved analytically, since equation (4.4) is transcendental. They are, however, well defined in the range $\frac{1}{4}\pi < 0 < \frac{1}{8}\pi$. We find that our numerical computations agree with our theory of multifurcations during this two-cycle. Additional features also become evident. For instance, we find the situation where a multifurcation leads to two different four-cycles instead of one eight-cycle. This happens at $x_1 = 0.4948\ldots$ with $v = 0.316\,63\ldots$ when $N = 4$. This unusual multifurcation is due to the fact that the original two-cycle FPs become

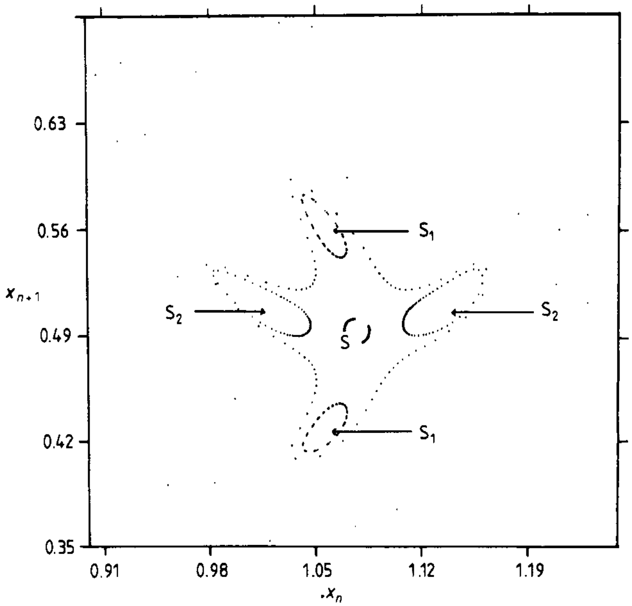


**Figure 8.** The 'starfish'. Orbits of the mapping at $v = 0.32$ after an anomalous $(2\times 2)$-furcation of one of the two-cycle FPs. The two different cycles are indicated by $S_1$ and $S_2$.

marginally stable at this point, that is $|\mathrm{Tr}\,\mathbf{M}| = 2$ also for the two-cycle mother orbit. For the ratios $2k/N$ in equation (4.5) we obtain $2k/N = 0$, which might give the real bifurcation. Figure 8 illustrates this curious phenomenon called 'starfish' by Bak and Jensen (1982). The other special multifurcation can be expected when $\mathrm{Tr}\,\mathbf{M} = -1$ for the mother orbit. This happens at $x_1 = 0.573\,34\ldots$ with $v = 0.2827\ldots$ when $N = 3$. Again, as in the case of the one-cycle mother orbit, we find that two different three-furcations take place at that point. The orbits after the three-furcations are shown in figure 9.

In the area of $(2\times 2)$-cycle orbits we can take advantage of the symmetry of the mapping (1.1), which states that $x_1 = \frac{1}{4}\pi + x_2$ in the whole range $\frac{1}{8}\pi < v < \frac{1}{8}(\pi^2+1)^{1/2}$. This gives us the possibility to predict multifurcations analytically. It is easy to see that $N$-furcations will happen for

$$v = \tfrac{1}{8}\{\pi^2 + \tfrac{1}{2}[1-\cos(2k\pi/N)]\}^{1/2}. \tag{4.6}$$

For bifurcations $2k/N = 0$ or 1. We find that the multifurcations given by equation (4.6) fit our numerical and graphical investigations surprisingly well irrespective of the fact that the multifurcation can sometimes lead only to the birth of unstable FPs. Such a situation happens for $2k/N = \frac{2}{3}$ (equation (4.6)). Each of the three-furcated FPs emerges as an unstable (hyperbolic with reflection) one. However, our theory predicts quite well the area where these unstable FPs turn to be stable. This is due to the fact that stabilisation takes place in the very vicinity of the mother orbit FPs.

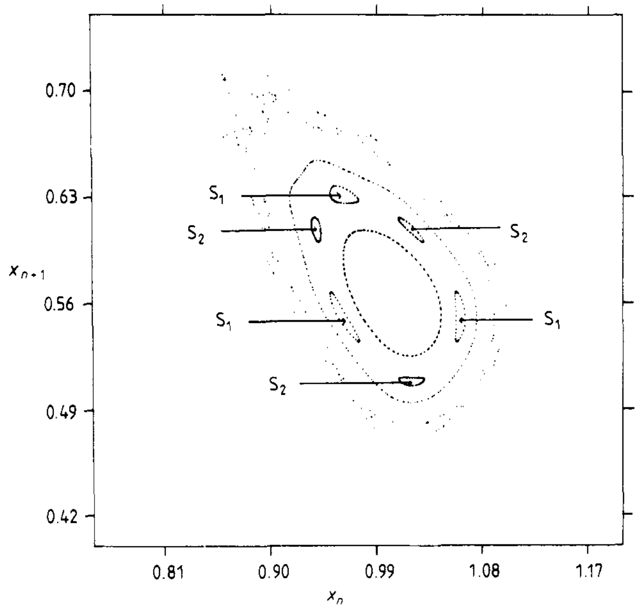


**Figure 9.** Orbits near the two-cycle FP at $v = 0.29$ after an anomalous $(2\times 3)$-furcation to two different six-cycles indicated by $S_1$ and $S_2$.

## 5. Conclusions

In the mappings, which describe real physical systems, the values of control parameters leading to ordinary bifurcations are often unphysically high. The attempts to study the chaotic behaviour of these systems through the ordinary bifurcations would be fruitless but, on the other hand, the values of control parameters of multifurcations are often in the range of physical interest. We make the conclusion that multifurcations in many of the mappings describing conservative physical systems form a dominant way in creating chaotic behaviour. This paper reports a number of analytical and numerical results for multifurcations in conservative systems. Their appearance is governed by a simple equation involving the Chebyshev polynomials.

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